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HEAVY-LIGHT MESONS IN THE SYMMETRIES OF EXTENDED NONRELATIVISTIC QUARK MODEL

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ABSTRACT

In present work, the modified Schrödinger equation (MSE) is analytically solved. The Heavy-Light Mesons (HLM) under modified nonrelativistic quark-antiquark potential, are extended to the symmetries of noncommutative quantum mechanics (NCQM), using the generalized Bopp's shift method. The energy a spectrum of HLM has been investigated in the framework of the perturbative quantum chromodynamics (PQCD) extended nonrelativistic quark model. The new energy eigenvalues and the corresponding Hamiltonian operator are calculated in the 3-dimensional noncommutative real space phase (NC: 3D-RSP) symmetries. The masses of the scalar, vector, pseudoscalar, and pseudovector for (B, B_s , D and D_s) mesons have been calculated in (NC: 3D-RSP). Moreover, using the perturbation approach, we found that the perturbative solutions of discrete spectrum can be expressed by the parabolic cylinder functions function $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$, Gamma function $\Gamma(\nu)$, the discreet atomic quantum numbers *(j, l, s, m)* of the $Q\bar{Q}$ state and (the spin independent and spin dependent) parameters (a, b, g, h), in addition to noncommutativity parameters (Θ and $\bar{\theta}$). Furthermore, we have shown that the total complete degeneracy of new energy levels of HLM was changed to become equals to the new value $3n^2$ instead to the old values n^2 in ordinary quantum mechanics. Our obtained results are in good agreement with the already existing literature in NCQM.

Keywords: *Schrödinger equation, Heavy-Light Mesons, the nonrelativistic quark-antiquark potential, Bopp's shift method, noncommutative space phase and the Weyl Moyal star product.*

1. INTRODUCTION

T It is well known that the study of deferent properties of Heavy-Light Mesons (HLM) has attracted attention from researchers specializing in this field, and in particular, the mass spectra of quarkonium systems as charmonium and bottomonium mesons with the quark-antiquark interaction potential for example ($c\bar{c}, b\bar{b}, c\bar{s}, b\bar{s}, b\bar{u}$) and $c\bar{b}$, is very important for understanding the structure of hadrons and dynamics of heavy quarks [1-6]. The researchers Abu-Shady M. and Khokha E. M., were studied the HLM under the combination of vector and scalar potentials and obtained the mass spectra and energy spectra of HLM (B, B_s , D and D_s mesons) [6]. The main objective is to develop the research article and expanding it to the hug symmetry known by noncommutative quantum mechanics (NCQM) in order to achieve a more accurate physical vision so

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that this study becomes valid in the field of nanotechnology. On the other hand, to explore the possibility of creating new applications and more profound interpretations in the sub-atomics and Nano scales using new version the modified nonrelativistic quark-antiquark potential, this has the following form:

$$V_{hlm}(r) = \underbrace{ar^2 + br + \delta + \frac{g}{r} + \frac{h}{r^3}}_{\text{Ordinary-QM}} \rightarrow \underbrace{V_{hlm}(\hat{r}) = V_{hlm}(r) + \left\{\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right\}\vec{\mathbf{L}}\vec{\Theta}}_{NCQM} \qquad (1)$$

It is important to mention that the noncommutativity theory was introduced [7] [8]. The new structure of NCQM based to new canonical commutations relations in both Schrödinger and Heisenberg pictures ((SP) and (HP)), respectively, as follows (Throughout this paper, the natural units $c = \hbar = 1$ will be used) [9-13]:

$$\begin{aligned} \left[\hat{x}_\mu \overset{*}{,} \hat{p}_\nu\right] &= \left[\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)\right] = i\delta_{\mu\nu} \\ \left[\hat{x}_\mu \overset{*}{,} \hat{x}_\nu\right] &= \left[\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)\right] = i\theta_{\mu\nu} \\ \left[\hat{p}_\mu \overset{*}{,} \hat{p}_\nu\right] &= \left[\hat{p}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)\right] = i\bar{\theta}_{\mu\nu} \end{aligned} \qquad (2)$$

The indices $\mu, \nu \equiv 1,2,3$. This means that the principle of uncertainty for Heisenberg was generalized to include rather than both position and momenta $(\hat{x}_\mu, \hat{p}_\nu)$ only to include also two positions $(\hat{x}_\mu, \hat{x}_\nu)$ and two momenta are: $(\hat{p}_\mu, \hat{p}_\nu)$ at the same time.

The very small two parameters $\theta^{\mu\nu}$ and $\bar{\theta}^{\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes, parameters of noncommutativity and $(*)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions $(f,g)(x,p)$ to the new form $\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p}) \equiv (f * g)(x,p)$ in (NC: 3D-RSP) symmetries as follows [13-21]:

$$(fg)(x,p) \rightarrow (f * g)(x,p) = \begin{pmatrix} fg - \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f \, \partial_\nu^x g \\ -\frac{i}{2}\bar{\theta}^{\mu\nu}\partial_\mu^p f \, \partial_\nu^p g \end{pmatrix}(x,p) \qquad (3)$$

The second and the third terms in the above equation are present the effects of (space-space) and (phase-phase) noncommutativity properties. However, the new operators $\hat{\xi}(t) = \hat{x}_\mu(t) \vee \hat{p}_\mu(t)$ in HP are depending to the corresponding new operators $\hat{\xi} = \hat{x}_\mu \vee \hat{p}_\nu$ in SP from the following projections relations:

$$\xi(t) = \exp(i\hat{H}_{hlm}(t-t_0))\xi\exp(-i\hat{H}_{hlm}(t-t_0)) \Rightarrow \hat{\xi}(t) = \exp(i\hat{H}_{nc-hlm}(t-t_0)) * \hat{\xi} * \exp(-i\hat{H}_{nc-hlm}(t-t_0)) \qquad (4)$$

Here $\xi = (x_\mu \vee p_\nu)$ and $\xi(t) = (x_\mu(t) \vee p_\nu(t))$, while the dynamics of new systems $\frac{\mathrm{d}\xi(t)}{\mathrm{dt}}$ are described from the following motion equations in NCQM

$$\frac{\mathrm{d}\xi(t)}{\mathrm{dt}} = \left[\xi(t), \hat{H}_{hlm}\right] \Rightarrow \frac{\mathrm{d}\hat{\xi}(t)}{\mathrm{dt}} = \left[\hat{\xi}(t) \overset{*}{,} \hat{H}_{nc-hlm}\right] \qquad (5)$$

Here $\hat{H}_{hlm}$ and $\hat{H}_{nc-hlm}$ are the quantum Hamiltonian operators for the HLM in the quantum mechanics and its extension, respectively. This paper consists of five sections and the organization scheme is given as follows: In the next section, we briefly review the ordinary SE with nonrelativistic quark-antiquark potential on based ref. [6]. Section 3 is devoted to studying the MSE by applying the generalized Bopp's shift method to obtain the modified nonrelativistic quark-antiquark potential and the modified spin-orbital operator. Then, we apply the standard perturbation theory to find the quantum spectrum of the generalized n^{th} excited state which produces by the effects of modified spin-orbital and modified Zeeman interactions. After that, in the fourth section, a discussion of the main results is present in addition to determining the new formula of mass spectra of the of HLM (B, B_s, D and D_s mesons) in (NC: 3D-RSP) symmetries. Finally, in section 5 we give concluding remark.

2. OVERVIEW OF THE HEAVY-LIGHT MESONS IN THE NONRELATIVISTIC QUARK MODEL IN COMMUTATIVE QUANTUM MECHANICS

In this section, we shall review the eigenvalues and eigenfunctions for the nonrelativistic quark-antiquark potential $V_{hlm}(r)$ [6]:

$$V_{hlm}(r) = ar^2 + br + \delta + \frac{g}{r} + \frac{h}{r^3} \tag{6}$$

Where $a\rangle$ and b are the spin-independent parameters of potential while (δ, g, h) are the spin dependent parameters of potential which given explicitly by equations (22), (23) and (24) in the main reference [6]. The complex eigen functions $\Psi(r,\theta,\varphi) = R(r)Y_l^m(\theta,\varphi)$ while the radial part $R_{nl}(r)$ satisfies the following differential equation [6]:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + 2\mu\left(E_{nl} - ar^2 - br - \delta - \frac{g}{r} - \frac{h}{r^3}\right)\right]R_{nl}(r) = 0 \tag{7}$$

Where l and E_{nl} represent angular momentum and energy and the magnetic number Check known inequality ($-l \le m \le +l$). The reduced mass of the quark and anti-quark system is $\mu = \frac{m_q \overline{m}_q}{m_q + \overline{m}_q}$. The complete normalized wave functions $\Psi_{n,l,m}(\overleftrightarrow{r})$ and corresponding energies $E_{n,l}$, respectively as follows [6]:

$$\Psi_{n,l,m}(\overleftrightarrow{r}) = \frac{C_{n,l}}{n!} r^{l+n} \exp\left(-\sqrt{\frac{\mu a}{2}} r^2 - \sqrt{\frac{\mu}{2a}} br\right) Y_l^m(\theta,\varphi) \tag{8}$$

$$E_{n,l} = \sqrt{\frac{a}{2\mu}}(2n+l+3) - \frac{b^2}{4a} + \delta - \frac{8h}{\nu^3} - \frac{h}{a}\left(\frac{9h}{4\nu^8} + \frac{3b}{2\nu^4}\right)$$

Here $C_{n,l} = n!\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)} \exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}^{1/2}$ is the normalization constant while $\alpha = \sqrt{\frac{\mu a}{2}}$ and $\beta = \frac{\mu b}{2\alpha}$. The parameter $\nu = 1/r_0$ (r_0 is a characteristic radius of the meson). It should be noted that the normalization constant has been quoted from the reference [3], because it treats the same wave function.

3. SOLUTION OF MODIFIED SCHRÖDINGER EQUATION FOR HLM UNDER MODIFIED NONRELATIVISTIC QUARK-ANTIQUARK POTENTIAL

3.1 REVIEW OF GENERALIZED BOPP'S SHIFT METHOD

In this sub-section, we shall give an overview or a brief preliminary for HLM under modified nonrelativistic quark-antiquark potential in (NC: 3D-RSP) symmetries. To perform this task the physical form of modified Schrödinger equation (MSE), it is necessary to replacing ordinary 3-dimensional Hamiltonian operator $\hat{H}(x_\mu, p_\mu)$, ordinary complex wave function $\Psi(\vec{r})$ and ordinary energy E_{nl} by new Hamiltonian operator $\hat{H}_{nc-hlm}(\hat{x}_\mu, \hat{p}_\mu)$, new complex wave function $\hat{\Psi}(\overleftrightarrow{r})$ and new values E_{nc-hlm}, respectively. In addition to replace the ordinary product by the Weyl Moyal star product, which allows us to construct the MSE in (NC-3D: RSP) symmetries as follows [22-25]:

$$\hat{H}_{hlm}(x_\mu, p_\mu)\Psi(\vec{r}) = E_{nl}\Psi(\vec{r}) \Rightarrow$$
$$\hat{H}(\hat{x}_\mu, \hat{p}_\mu) * \Psi(\vec{\hat{r}}) = E_{nc-hlm}\Psi(\vec{\hat{r}}) \qquad (9)$$

The Bopp's shift method has been successfully applied to relativistic and nonrelativistic noncommutative quantum mechanical problems using modified Dirac equation, modified Klein-Gordon equation and MSE, respectively. This method has produced very promising results for a number of situations having a physical, chemical interest. The method reduces to the Dirac equation, Klein-Gordon and equation Schrödinger under two simultaneously translations in space and phase. It based on the following new commutators [26-30]:

$$[\hat{x}_\mu, \hat{x}_\nu] = [\hat{x}_\mu(t), \hat{x}_\nu(t)] = i\theta_{\mu\nu}$$
$$[\hat{p}_\mu, \hat{p}_\nu] = [\hat{p}_\mu(t), \hat{p}_\nu(t)] = i\bar{\theta}_{\mu\nu} \qquad (10)$$

The new generalized positions and momentum coordinates $(\hat{x}_\mu, \hat{p}_\nu)$ in (NC: 3D-RSP) are defined in terms of the commutative counterparts (x_μ, p_ν) in ordinary quantum mechanics via, respectively [30-33]:

$$(x_\mu, p_\nu) \Rightarrow (\hat{x}_\mu, \hat{p}_\nu) = \left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu,\ p_\mu + \frac{\bar{\theta}_{\mu\nu}}{2}x_\nu\right) \qquad (11)$$

The above equation allows us to obtain the two operators $(\hat{r}^2, \hat{p}^2)$ in (NC-3D: RSP) symmetries as follows [30-35]:

$$(r^2, p^2) \Rightarrow (\hat{r}^2, \hat{p}^2) = \left(r^2 - \vec{\mathbf{L}}\vec{\Theta},\ p^2 + \vec{\mathbf{L}}\vec{\bar{\theta}}\right) \qquad (12)$$

The new two couplings $\mathbf{L}\Theta$ and $\vec{\mathbf{L}}\vec{\bar{\theta}}$ are $(L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13})$ and $(L_x\bar{\theta}_{12} + L_y\bar{\theta}_{23} + L_z\bar{\theta}_{13})$, respectively and ($L_x$, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ while $\Theta_{\mu\nu} = \theta_{\mu\nu}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\hat{H}(\hat{x}_\mu, \hat{p}_\mu) * \Psi(\vec{\hat{r}}) = E_{nc-hlm}\Psi(\vec{\hat{r}})$$
$$\Rightarrow H(\hat{x}_\mu, \hat{p}_\mu)\psi(\vec{r}) = E_{nc-hlm}\psi(\vec{r}) \qquad (13)$$

The new Hamiltonian operator $H_{nc-hlm}(\hat{x}_\mu, \hat{p}_\nu)$ of HLM can be expressed as:

$$H_{hlm}(x_\mu, x_\mu) \Rightarrow H_{nc-hlm}(\hat{x}_\mu, \hat{p}_\mu) \equiv H\left(\hat{x}_\mu = x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu, \hat{p}_\mu = p_\mu + \frac{\bar{\theta}_{\mu\nu}}{2}x_\nu\right) = \frac{\hat{p}^2}{2\mu} + V_{hlm}\left(\hat{r} = \sqrt{\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x_\mu - \frac{\theta_{\mu\alpha}}{2}p_\alpha\right)}\right) \qquad (14)$$

Where $V_{hlm}(\hat{r})$ denote to the modified nonrelativistic quark-antiquark potential in (NC: 3D-RSP) symmetries:

$$V_{hlm}(r) \Rightarrow V_{hlm}(\hat{r}) = a\hat{r}^2 + b\hat{r} + \delta + \frac{g}{\hat{r}} + \frac{h}{\hat{r}^3} \qquad (15)$$

Again, applying Eq. (12) to obtain the four terms ($a\hat{r}^2, b\hat{r}, \frac{g}{\hat{r}}$ and $\frac{h}{\hat{r}^3}$) that we will use to find the modified nonrelativistic quark-antiquark potential $V_{hlm}(\hat{r})$, and after straightforward calculations, we can find the following terms in (NC: 3D-RSP) symmetries as:

$$ar^2 \rightarrow a\hat{r}^2 = ar^2 - a\vec{\mathbf{L}}\vec{\Theta} + O(\Theta^2)$$
$$br \rightarrow b\hat{r} = br - \frac{b}{2r}\vec{\mathbf{L}}\vec{\Theta} + O(\Theta^2)$$
$$\frac{g}{r} \rightarrow \frac{g}{\hat{r}} = \frac{g}{r} + \frac{g}{2r^3}\vec{\mathbf{L}}\vec{\Theta} + O(\Theta^2)$$
$$\frac{h}{r^3} \rightarrow \frac{h}{\hat{r}^3} = \frac{h}{r^3} + \frac{3h}{2r^5}\vec{\mathbf{L}}\vec{\Theta} + O(\Theta^2) \qquad (16)$$

Substituting, Eq. (16) into Eq. (15), gives the modified nonrelativistic quark-antiquark potential in (NC-3D: RSP) symmetries as follows:

$$V_{hlm}(\hat{r}) = V_{hlm}(r) + \left(\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right)\vec{\mathbf{L}}\vec{\Theta} \qquad (17)$$

By making the substitution above equation into Eq. (14), we find the global our working new modified Hamiltonian operator $H_{\text{nc-}h\text{lm}}(\hat{r})$ in (NC: 3D-RSP) symmetries:

$$H_{hlm}(x_\mu, p_\nu) \Rightarrow H_{nc-hlm}(\hat{r}) = H_{hlm}(x_\mu, p_\nu) + H_{\text{per-}h\text{lm}}(r) \qquad (18)$$

Here $H_{hlm}(x_\mu, p_\nu)$ is just the ordinary Hamiltonian operator with nonrelativistic quark-antiquark potential in commutative quantum mechanics:

$$H_{hlm}(x_\mu, p_\mu) = \frac{p^2}{2\mu} + ar^2 + br + \delta + \frac{g}{r} + \frac{h}{r^3} \qquad (19)$$

While the rest part $H_{\text{per-}h\text{lm}}(r)$ in Eq. (18) is proportional with two infinitesimals parameters (Θ and $\bar{\theta}$) and its analytical expression is as follows:

$$H_{\text{per-}h\text{lm}}(r) = \left(\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right)\mathbf{L}\vec{\Theta} + \frac{\vec{\mathbf{L}}\vec{\vec{\boldsymbol{\theta}}}}{2\mu} \qquad (20)$$

Thus, we can consider $H_{\text{per-}h\text{lm}}(r)$ as a perturbation term compared with the principal Hamiltonian operator. $H_{hlm}(x_\mu, p_\mu)$

3.2 THE EXACT MODIFIED SPIN-ORBIT SPECTRUM FOR HEAVY-LIGHT MESONS UNDER MODIFIED NONRELATIVISTIC QUARK-ANTIQUARK POTENTIAL IN GLOBAL (NC: 3D- RSP) SYMMETRIES

In this subsection, we will apply the same strategy, which we have seen exclusively in some of our published scientific works. Under such a particular choice, one can easily reproduce both couplings ($\vec{\mathbf{L}}\vec{\Theta}$ and $\vec{\mathbf{L}}\vec{\vec{\theta}}$) to the new physical forms ($g_s\Theta\vec{L}\vec{S}$ and $g_s\bar{\theta}\vec{L}\vec{S}$), respectively. Thus, the new forms of $H_{\text{so-}h\text{lm}}(r,\Theta,\bar{\theta})$ for Heavy-Light Mesons under modified nonrelativistic quark-antiquark potential as follows [32-35]:

$$H_{\text{per-hlm}}(r) \to H_{\text{so-hlm}}(r,\Theta,\bar{\theta}) \equiv g_s\left\{\left(\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right) + \frac{\bar{\theta}}{2\mu}\right\}\vec{L}\vec{S} \quad (21)$$

Here $\Theta = \sqrt{\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2}$, $\bar{\theta} = \sqrt{\bar{\theta}_{12}^2 + \bar{\theta}_{23}^2 + \bar{\theta}_{13}^2}$ and g_s is a new constant, which plays the role of the strong coupling constant in the quantum chromodynamics theory QCD, we have chosen the two vectors $\vec{\Theta}$ and $\vec{\vec{\boldsymbol{\theta}}}$ parallel to the spin $\vec{S}$ of Heavy-Light Mesons. Furthermore, the above perturbative terms $H_{\text{per-}h\text{lm}}(r)$ can be rewritten to the following new form:

$$H_{so-hlm}(r,\Theta,\bar{\theta}) = \frac{g_s}{2}\left\{\left(\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right)\Theta + \frac{\bar{\theta}}{2\mu}\right\}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right) \quad (22)$$

Where $\vec{J}$ and $\vec{S}$ are defined the operators of the total angular momentum and spin of Heavy-Light Mesons such as B , B_s , D and D_s mesons. This operator traduces the coupling between spin $\vec{S}$ and orbital momentum $\vec{L}$. The set ($H_{so-hlm}(r,\Theta,\bar{\theta})$, J^2, L^2 , S^2 and J_z) forms a complete of conserved physics quantities and for $\overleftrightarrow{S} = \vec{1}$, the eigenvalues of the spin-orbit coupling operator are $k(l) \equiv \frac{1}{2}\{j(j+1) - l(l+1) - 2\}$ corresponding $j = l + 1$ (spin great), $j = l$ (spin middle) and $j = l - 1$ (spin little), respectively. Then, one can form a diagonal (3×3) matrix for modified nonrelativistic quark-antiquark potential with diagonal element $(H_{so-hlm})_{11}$, $(H_{so-hlm})_{22}$ and $(H_{so-hlm})_{33}$ in (NC: 3D-RSP) symmetries:

$$\begin{aligned}
(H_{so-hlm})_{11} &= g_s k_1(l)\left(\left(\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right)\Theta + \frac{\bar{\theta}}{2\mu}\right) \text{if } j = l+1 \\
(H_{so-hlm})_{22} &= g_s k_2(l)\left(\left(\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right)\Theta + \frac{\bar{\theta}}{2\mu}\right) \text{if } j = l \\
(H_{so-hlm})_{33} &= g_s k_3(l)\left(\left(\frac{h}{2r^5} + \frac{g}{2r^3} - \frac{b}{2r} - a\right)\Theta + \frac{\bar{\theta}}{2\mu}\right) \text{if } j = l-1
\end{aligned} \qquad (23)$$

Here $(k_1(l), k_2(l), k_3(l)) \equiv \frac{1}{2}(l, -2, -2l-2)$. The non-null diagonal elements $(H_{so-hlm})_{11}$, $(H_{so-hlm})_{22}$ and $(H_{so-hlm})_{33}$ of the modified Hamiltonian operator $H_{nc-hlm}(\hat{r})$ can be influence to the energy values E_{nl} by creating three new values:

$$\begin{cases}
E_{\text{g-hlm}} = \langle\Psi(r,\theta,\varphi)|(H_{so-hlm})_{11}|\Psi(r,\theta,\varphi)\rangle \\
E_{\text{m-hlm}} = \langle\Psi(r,\theta,\varphi)|(H_{so-hlm})_{22}|\Psi(r,\theta,\varphi)\rangle \\
E_{\text{l-hlm}} = \langle\Psi(r,\theta,\varphi)|(H_{so-hlm})_{33}|\Psi(r,\theta,\varphi)\rangle
\end{cases} \quad (24)$$

We will see them in detail in the next subsection. After the profound calculation, one can show that the new radial function $R_{nl}(r)$ satisfying the following differential equation for Heavy-Light Mesons under modified nonrelativistic quark-antiquark potential:

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+2\mu\left(\begin{array}{l}E_{nc-hlm}-ar^2-br-\delta-\frac{g}{r}-\frac{h}{r^3}\\-\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\vec{\mathbf{L}}\vec{\Theta}-\frac{\vec{\mathbf{L}}\vec{\bar{\theta}}}{2\mu}\end{array}\right)\right]R_{nl}(r)=0 \quad (25)$$

Through our investigation to the expression of the perturbative part $H_{\text{per-}hlm}(r)$ that we saw in the Eq. (20) we find it proportionate to two infinitesimals parameters (Θ and $\bar{\theta}$). Thus in what follows, we proceed to solve the modified radial part of the MSE that is Eq. (25) by applying standard perturbation theory to find acceptable solutions at first order of two parameters Θ and $\bar{\theta}$. The proposed solutions for MSE under modified nonrelativistic quark-antiquark potential includes energy corrections, which product automatically from two principal physical phoneme's, the first one is the effect of the modified spin-orbit interaction and the second is the modified Zeeman effect while the stark effect can be appear in the linear part of modified nonrelativistic quark-antiquark potential.

3.3 THE EXACT MODIFIED SPIN-ORBIT SPECTRUM FOR HEAVY-LIGHT MESONS UNDER A MODIFIED NONRELATIVISTIC QUARK-ANTIQUARK MODEL IN (NC: 3D-RSP) SYMMETRIES

The purpose here is to give a complete prescription for determine the energy level of n^{th} excited state, for HLM such as scalar, vector, pseudoscalar, and pseudovector for (B , B_s , D and D_s) mesons under modified nonrelativistic quark-antiquark potential. We first find the corrections ($E_{g-hlm}(k_1(l),a,b,g,h,n)$, $E_{m-hlm}(k_2(l),a,b,g,h,n)$ and $E_{l-hlm}(k_3(l),a,b,g,h,n)$) which are generated with the non-null diagonal elements $(H_{so-hlm})_{11}$, $(H_{so-hlm})_{22}$ and $(H_{so-hlm})_{33}$ corresponding $j=l+1$ (spin great), $j=l$ (spin middle) and $j=l-1$ (spin little), respectively, at first order of two parameters (Θ and $\bar{\theta}$). Moreover, by applying the perturbative quantum chromodynamics (PQCD) nonrelativistic quark model, we obtained the following results:

$$\begin{aligned}E_{g-hlm}&=g_s\frac{C_{n,l}{}^2}{n!^2}k_1(l)\int_0^{+\infty}r^{2l+2n+2}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2\sqrt{\frac{\mu}{2a}}br\right)\left(\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)dr\\E_{m-hlm}&=g_s\frac{C_{n,l}{}^2}{n!^2}k_2(l)\int_0^{+\infty}r^{2l+2n+2}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2\sqrt{\frac{\mu}{2a}}br\right)\left(\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)dr\\E_{l-hlm}&=g_s\frac{C_{n,l}{}^2}{n!^2}k_3(l)\int_0^{+\infty}r^{2l+2n+2}\exp\left(-2\sqrt{\frac{\mu a}{2}}r^2-2\sqrt{\frac{\mu}{2a}}br\right)\left(\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)dr\end{aligned} \quad (26)$$

We have used the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi=\delta_{ll'}\delta_{mm'}$. Now, we can rewrite the above three equations to the new simplified form:

$$\begin{aligned}E_{g-hlm}(k_1(l),a,b,g,h,n)&=g_s\frac{C_{n,l}{}^2}{n!^2}k_1(l)\left\{\Theta[T_1(h,n,a,b)+T_2(g,n,a,b)+T_3(b,n,a)+T_4(a,n,b)]+\frac{\bar{\theta}}{2\mu}T_5(n,a,b)\right\}\\E_{m-hlm}(k_2(l),a,b,g,h,n)&=g_s\frac{C_{n,l}{}^2}{n!^2}k_2(l)\left\{\Theta[T_1(h,n,a,b)+T_2(g,n,a,b)+T_3(b,n,a)+T_4(a,n,b)]+\frac{\bar{\theta}}{2\mu}T_5(n,a,b)\right\}\\E_{l-hlm}(k_3(l),a,b,g,h,n)&=g_s\frac{C_{n,l}{}^2}{n!^2}k_3(l)\left\{\Theta[T_1(h,n,a,b)+T_2(g,n,a,b)+T_3(b,n,a)+T_4(a,n,b)]+\frac{\bar{\theta}}{2\mu}T_5(n,a,b)\right\}\end{aligned} \quad (27)$$

Moreover, the expressions of the 5-factors $T_i\left(i=\overline{1,5}\right)$ are given by:

$$T_1(h,n,a,b)=\frac{h}{2}\int_0^{+\infty} r^{2l+2n-2-1}\exp\left(-\lambda r^2-\gamma r\right)dr,$$
$$T_2(g,n,a,b)=\frac{g}{2}\int_0^{+\infty} r^{2l+2n-1}\exp\left(-\lambda r^2-\gamma r\right)dr \qquad (28)$$
$$T_3(b,n,a)=-\frac{b}{2}\int_0^{+\infty} r^{2l+2n+2-1}\exp\left(-\lambda r^2-\gamma r\right)dr$$
$$\text{and } T_4(a,n,a)=-aT_5(n,a,b)=-a\int_0^{+\infty} r^{2l+2n+3-1}\exp\left(-\lambda r^2-\gamma r\right)dr$$

Where $\lambda=\sqrt{2\mu a}$ and $\gamma=\sqrt{\frac{2\mu}{a}b}$. Now, we apply the following special integration [36]:

$$\int_0^{+\infty} x^{\nu-1}\exp\left(-\lambda x^2-\gamma x\right)dx=(2\lambda)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right) \qquad (29)$$

Where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$ and $\Gamma(\nu)$ denote to the Parabolic cylinder functions and Gamma function, respectively. $\mathrm{Rel}(\lambda)\rangle 0$ and $\mathrm{Rel}(\nu\rangle 0)$.
After straightforward calculations, we can obtain the explicitly results:

$$T_1(h,n,a,b)=\frac{h}{2}(2\lambda)^{-\frac{2l+2n-2}{2}}\Gamma(2l+2n-2)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-(2l+2n-2)}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$$
$$T_2(g,n,a,b)=\frac{g}{2}(2\lambda)^{-\frac{2l+2n-1}{2}}\Gamma(2l+2n-1)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-(2l+2n-1)}\left(\frac{\gamma}{\sqrt{2\lambda}}\right) \qquad (30)$$
$$T_3(b,n,a)=-\frac{b}{2}(2\lambda)^{-\frac{2l+2n+2}{2}}\Gamma(2l+2n+2)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-(2l+2n+2)}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$$
$$T_4(a,n,a)=-aT_5(n,a,b)=-a(2\lambda)^{-\frac{2l+2n+3}{2}}\Gamma(2l+2n+3)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-(2l+2n-2)+3}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$$

Allow us to obtain the exact modifications ($E_{g-hlm}(k_1(l),a,b,g,h,n)$, $E_{m-hlm}(k_2(l),a,b,g,h,n)$ and $E_{l-hlm}(k_3(l),a,b,g,h,n)$) for Heavy-Light Mesons under modified nonrelativistic quark-antiquark potential as:

$$E_{g-hlm}(k_1(l),a,b,g,h,n)=g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}k_1(l)\left\{\Theta T(a,b,g,h,n)+\frac{\bar{\theta}}{2\mu}T_5(n,a,b)\right\}$$
$$E_{m-hlm}(k_2(l),a,b,g,h,n)=g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}k_2(l)\left\{\Theta T(a,b,g,h,n)+\frac{\bar{\theta}}{2\mu}T_5(n,a,b)\right\} \qquad (31)$$
$$E_{l-hlm}(k_3(l),a,b,g,h,n)=g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}k_3(l)\left\{\Theta T(a,b,g,h,n)+\frac{\bar{\theta}}{2\mu}T_5(n,a,b)\right\}$$

With;
$T(a,b,g,h,n)=T_1(h,n,a,b)+T_2(g,n,a,b)+T_3(b,n,a)+T_4(a,n,b)$.

3.4 The exact modified magnetic spectrum for Heavy-Light Mesons under modified nonrelativistic quark-antiquark potential in (NC: 3D- RSP) symmetries

Further to the important previously obtained results, now, we consider other important physically meaningful phenomena induced with self-uniform magnetic field $\vec{B}$. This field is self-generated from the properties of (space-space) and (phase-phase) noncommutativity influenced on HLM such as scalar, vector, pseudoscalar, and pseudovector for (B , B_s , D and D_s) mesons, under modified nonrelativistic quark-antiquark potential related to the influence of an external. To avoid the repetition in the theoretical calculations, it is sufficient to apply the two following simultaneously replacements:

$$\begin{cases}\vec{\Theta}\to\chi\vec{B}\\ \vec{\bar{\theta}}\to\bar{\sigma}\vec{B}\end{cases}\Rightarrow\left(\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)\text{will - be -}$$
$$\text{replace - by}:\left(\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\vec{B}\vec{L} \tag{32}$$

Here χ and $\bar{\sigma}$ are two infinitesimal real proportional constants.

It should be noted that the generation of this magnetic phenomenon reflects one of the most important features of the noncommutativity properties of (space-space) and (phase-phase). Moreover, if we choose the generated magnetic field $\vec{B}$ parallel to the (Oz) axis, which allows us to introduce the new modified magnetic Hamiltonian $H_{m-hlm}(r,\chi,\bar{\sigma})$ in (NC: 3D-RSP) symmetries as follows:

$$H_{\text{so-hlm}}(r,\Theta,\bar{\theta})\to H_{m-hlm}(r,\chi,\bar{\sigma})$$
$$=\left(\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\left\{\vec{B}\vec{J}-\aleph_z\right\} \tag{33}$$

Here $\aleph_z\equiv-\vec{S}\vec{B}$ denote to Zeeman Effect in commutative quantum mechanics, while $\aleph_{\text{mod}-z}\equiv\vec{B}\vec{J}-\aleph_z$ is the new Zeeman effect in (NC: 3D-RSP) symmetries. To obtain the exact NC magnetic modifications of energy for n^{th} excited states of Heavy-Light Mesons, $E_{mag-hlm}(m,n,a,b,g,h)$ we just replace the two parameters $k_1(l)$ and Θ in the Eq. (35) by the corresponding quantum parameters m and χ , respectively:

$$E_{mag-hlm}(m,n,a,b,g,h)=g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}\left\{\chi T(a,b,g,h,n)+\frac{\bar{\sigma}}{2\mu}T_5(n,a,b)\right\}Bm \tag{34}$$

We have $-l\le m\le+l$, which allows us to fix ($2l+1$) values for discreet numbers m . It should be noted that the results obtained in Eq. (34) could find it by direct calculation $E_{mag-hlm}=\langle\Psi(r,\theta,\varphi)|H_{m-hlm}(r,\chi,\bar{\sigma})|\Psi(r,\theta,\varphi)\rangle$ that takes the following explicit relation:

$$E_{mag-hlm}=g_s\frac{C_{n,l}^{\;2}}{n!^2}k_1(l)\int_0^{+\infty}r^{2l+2n+2}\exp\left(-\sqrt{2\mu a}r^2-\sqrt{\frac{2\mu}{a}}br\right)\left(\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)dr \tag{35}$$

Then we find the corrections produced by the operator $H_{m-hlm}(r,\chi,\bar{\sigma})$ for n^{th} excited states repeating the same calculations in the previous subsection. As we have already indicated at the beginning of this sub-section that the genera rated phenomenon of magnetic field is automatic due to the new properties of space-space resulting from the new algebra that we saw in Eq.(2), this is what we discussed in our previous studies [37-40].

4. MAIN RESULTS

In the previous sections from our current research, we obtained the solution of the modified

Schrödinger equation for HLM such as scalar, vector, pseudoscalar, and pseudovector for (B , B_s , D and D_s) mesons under modified nonrelativistic quark-antiquark potential, which is given in Eq. (17) by using the generalized Bopp's shift method and standard perturbation theory. The energy eigenvalue is calculated in the noncommutative three-dimensional real space and phase. Now, using the superposition principle, we can obtain the eigen energies $\left(E_{\text{nc-g}hlm}, E_{\text{nc-m}hlm}, E_{\text{nc-l}hlm}\right)(m,n,j,l,a,b,g,h)$ for MSE for heavy-light mesons with spin-1 on based on our original results, which presented on the Eqs. (31) and (34) in addition to the ordinary energy E_{nl} for nonrelativistic quark-antiquark potential presented in the Eq. (8) as follows:

$$E_{\text{nc-g}hlm}(m,n,j=l+1,l,a,b,g,h)=E_{nl}+g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}\left\{(k_1(l)\Theta+\chi Bm)T(n,a,b,g,h)-\left(\frac{\bar{\theta}}{2\mu}k_1(l)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_5(n,a,b)\right\}$$
$$E_{\text{nc-m}hlm}(m,n,j=l,l,a,b,g,h)=E_{nl}+g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}\left\{(k_2(l)\Theta+\chi Bm)T(n,a,b,g,h)-\left(\frac{\bar{\theta}}{2\mu}k_1(l)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_5(n,a,b)\right\}$$
$$E_{\text{nc-lch}lm}(m,n,j=l-1,l,a,b,g,h)=E_{nl}+g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}\left\{(k_3(l)\Theta+\chi Bm)T(n,a,b,g,h)-\left(\frac{\bar{\theta}}{2\mu}k_1(l)+\frac{\bar{\sigma}}{2\mu}Bm\right)T_5(n,a,b)\right\} \quad (36)$$

This is one of the main objectives of our research and by noting that, the obtained eigenvalues of energies are real's and then the NC diagonal Hamiltonian $H_{nc-hlm}(x_\mu, p_\mu)$ is Hermitian, furthermore it's possible to write the three elements $(H_{nc-hlm})_{11}$, $(H_{nc-hlm})_{22}$ and $(H_{nc-hlm})_{33}$ as follows:

$$H_{hlm}(x_\mu,p_\mu)\rightarrow H_{nc-hlm}(x_\mu,p_\mu)\equiv\begin{pmatrix}(H_{nc-hlm})_{11} & 0 & 0\\ 0 & (H_{nc-hlm})_{22} & 0\\ 0 & 0 & (H_{nc-hlm})_{33}\end{pmatrix} \quad (37)$$

Where $(H_{nc-hlm})_{11}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-ghlm}$, $(H_{nc-hlm})_{22}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-mhlm}$ and $(H_{nc-cp})_{33}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-lhlm}$ with $\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\overrightarrow{\overrightarrow{\theta}}\vec{L}-\overrightarrow{\overrightarrow{\sigma}}\vec{L}}{2\mu}$ and the three modified interactions elements $\left(H_{\text{int}-ghlm}, H_{\text{int}-mhlm}, H_{\text{int}-lhlm}\right)$ are given by:

$$V_{hlm}(r)\rightarrow\begin{cases}H_{\text{int}-ghlm}=ar^2+br+\delta+\frac{g}{r}+\frac{h}{r^3}+g_s\left(k_1(l)\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\\ H_{\text{int}-mhlm}=ar^2+br+\delta+\frac{g}{r}+\frac{h}{r^3}+g_s\left(k_2(l)\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\\ H_{\text{int}-lhlm}=ar^2+br+\delta+\frac{g}{r}+\frac{h}{r^3}+g_s\left(k_3(l)\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{h}{2r^5}+\frac{g}{2r^3}-\frac{b}{2r}-a\right)\end{cases} \quad (38)$$

Thus, the ordinary kinetic term for nonrelativistic quark-antiquark potential ($-\frac{\Delta}{2\mu}$) and ordinary interaction $ar^2+br+\delta+\frac{g}{r}+\frac{h}{r^3}$, were replaced by a new modified form of kinetic term $\frac{\Delta_{nc}}{2\mu}$ and new modified interactions modified to the new form $\left(H_{\text{int}-ghlm}, H_{\text{int}-mhlm}, H_{\text{int}-lhlm}\right)$in (NC-3D: RSP) symmetries, respectively. On the other hand, it is evident to consider the quantum number m takes

($2l+1$) values and we have also two values for $(j=l\pm 1,l)$, thus every state in usually three-dimensional space of energy for heavy quarkonium system under modified nonrelativistic quark-antiquark potential will be $(3(2l+1))$ sub-states. To obtain the total complete degeneracy of energy level of the modified nonrelativistic quark-antiquark potential in (NC-3D: RSP) symmetries, we need to sum for all allowed values of l. Total degeneracy is thus,

$$\sum_{l=0}^{n-1}(2l+1)=n^2 \rightarrow 3\left(\sum_{l=0}^{n-1}(2l+1)\right)\equiv 3n^2 \qquad (39)$$

Now, we note that the obtained new energy eigenvalues $(E_{\text{nc-g}h\text{lm}},E_{\text{nc-m}h\text{lm}},E_{\text{nc-l}h\text{lm}})(m,n,j,l,a,b,g,h)$ now depend to new discrete atomic quantum numbers (n,j,l,s) and m in addition to the parameters (a,b,g,h) of the nonrelativistic quark-antiquark potential. It is pertinent to note that when the atoms have $\vec{S}=\vec{0}$, the total operator can be obtains from the interval $|l-s|\leq j\leq|l+s|$, which allow us to obtain the eigenvalues of the operator $\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$ as $k(j,l,s)\equiv 0$ and then the nonrelativistic energy spectrum $(E_{\text{nc-g}h\text{lm}},E_{\text{nc-m}h\text{lm}},E_{\text{nc-l}h\text{lm}})(m,n,j,l,a,b,g,h)$ reads:

$$(E_{\text{nc-g}h\text{lm}},E_{\text{nc-m}h\text{lm}},E_{\text{nc-l}h\text{lm}})(m,n,j,l,a,b,g,h)=E_{nl}+g_s\left\{\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\right\}\left\{\chi T(l,m,a,b,g,h)-\frac{\overline{\sigma}}{2}T_5(n,a,b)\right\}Bm \qquad (40)$$

Our last application is to calculate the modified mass spectra of the Heavy-Light Mesons such as scalar, vector, pseudoscalar, and pseudovector for (B , B_s , D and D_s) mesons under modified nonrelativistic quark-antiquark potential. In order to achieve this goal, we generalize the traditional formula $M=m_q+m_{\bar{q}}+E_{nl}$ to the new form:

$$M=m_q+m_{\bar{q}}+E_{nl}\rightarrow M_{nc-cp}=m_q+m_{\bar{q}}+\frac{1}{3}(E_{\text{nc-g}h\text{lm}}+E_{\text{nc-m}h\text{lm}}+E_{\text{nc-l}h\text{lm}})(m,n,j,l,a,b,g,h) \qquad (41)$$

Here $\frac{1}{3}(E_{\text{nc-g}h\text{lm}}+E_{\text{nc-m}h\text{lm}}+E_{\text{nc-l}h\text{lm}})(m,n,j,l,a,b,g,h)$ is the non-polarized energy value. Thus, the modified mass of Heavy-Light Mesons M_{nc-hlm} such as (B , B_s , D and D_s) mesons:

$$M_{nc-hlm}=M+g_s\frac{2(2\alpha)^{l+n+\frac{3}{2}}}{\Gamma\left(l+n+\frac{3}{2}\right)}\exp\left(-\frac{\beta^2}{2\alpha}\right)\begin{cases}\left\{\left(\chi Bm-\frac{l+4}{6}\Theta+\right)T(n,a,b,g,h)-\left(\frac{\overline{\sigma}}{2\mu}Bm-\frac{\overline{\theta}(l+4)}{12\mu}\right)T_5(n,a,b)\right\} & \text{for } \vec{S}=\vec{1}\\ \left\{\chi T(n,a,b,g,h)-\frac{\overline{\sigma}}{2}T_4(n,a,b)\right\}Bm & \text{for } \vec{S}=\vec{0}\end{cases} \qquad (42)$$

Here M is the Heavy-Light Mesons under nonrelativistic quark-antiquark potential in commutative quantum mechanics that was mentioned in the reference [6]. If we consider $(\Theta,\chi)\rightarrow(0,0)$, we recover the results of the commutative space of ref. [6], which means that our calculations are correct.

5. CONCLUSION

In the present work, the 3-dimensional modified Schrodinger equation is analytically solved using the generalized Bopp's shift method and standard perturbation theory. The nonrelativistic quark-antiquark potential is extended to include the effect of the noncommutativity space phase based on ref. [6]; we resume the main obtained results. Ordinary nonrelativistic quark-antiquark potential ($ar^2+br+\delta+\frac{g}{r}+\frac{h}{r^3}$) replaced by new modified interactions $(H_{\text{int}-ghlm},H_{\text{int}-mhlm},H_{\text{int}-lhlm})$ for Heavy-Light Mesons in (NC-3D: RSP) symmetries, The ordinary kinetic term $-\frac{\Delta}{2\mu}$ modified to the new form $\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\vec{\overline{\theta}}\vec{L}-\vec{\overline{\sigma}}\vec{L}}{2\mu}$ for Heavy-Light Mesons under influence of modified nonrelativistic quark-antiquark potential. We obtained the perturbative corrections $(E_{\text{nc-g}h\text{lm}},E_{\text{nc-m}h\text{lm}},E_{\text{nc-l}h\text{lm}})(m,n,j,l,a,b,g,h)$ for n^{th} excited state with (spin $\vec{S}=\vec{1}$ and $\vec{S}=\vec{0}$) for MSE for Heavy-Light Mesons under influence modified nonrelativistic quark-antiquark potential in (NC-3D: RSP) symmetries. We have obtained the mass spectra of heavy-light mesons (B , B_s , D and D_s mesons) in the extended non-relativistic quark model, the new values M_{nc-hlm} equal the sum of corresponding value M in CQM and two perturbative terms proportional with two parameters (Θ and $\overline{\theta}$). Through the high-value results, which we have achieved in present work, we hope to extend our recently work physics for further investigations of elementary particles and other characteristics of quarkonium among others. Our new treatment was more accurate glasses in which we delved into energy and mass values of HLM in the context of chromodynamics quantum mechanics symmetries.

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